Chapter Eight

Derivability of a real valued function

§ 8.1. DERIVATIVE AT A POINT

If a function f is defined on a *nbd* of a point c and

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \quad \text{or,} \quad \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

exists then f is said to be derivable at c, and this limit denoted by $f'(c)$ is called the derivative with respect to x at c of the function. The process of evaluating $f'(c)$ is called differentialtion. A function derivable at c is also said differentiable at c. For a single-valued function differentiability and derivability bear the same sense.

One sided derivability

A function f is said to be derivable from the right or left at a point c according as

$$\lim_{h \to 0+} \frac{f(c+h)-f(c)}{h} \quad \text{or,} \quad \lim_{h \to 0-} \frac{f(c+h)-f(c)}{h}$$

exists finitely. These limits are denoted by $f'(c+0)$ or $Rf'(c)$ and $f'(c-0)$ or $Lf'(c)$, called respectively the right derivative and left derivative.

If f is derivable at c, both $f'(c+0)$ and $f'(c-0)$ exist and are equal.

i.e., $Rf'(c) = L f'(c)$.

Note: If $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists infinitely, even then we shall admit that f is derivable at c. Unless otherwise stated, $f'(c)$ will be assumed to be finite.

Theorem 8.1.1: Derivability at a point c implies continuity at the point but not the converse.

Proof : Let f be derivable at c, then $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists finitely, and the limit is $f'(c)$.

$$\text{Now } \lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \times (x-c)$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \times \lim_{x \to c} (x-c) = f'(c) \times 0 = 0.$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f(x)$ is continuous at $x = c$.

In order to show that the converse is not true we consider the following example.

$f(x) = |x| \quad x \in \mathbb{R}$.

$|f(x) - f(0)| = ||x| - 0| = |x| < \varepsilon$, if $|x - 0| < \delta (= \varepsilon)$.

$\therefore f(x)$ is continuous at $x = 0$.

$$\text{Now, } \lim_{x \to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = 1.$$

$$\text{Also, } \lim_{x \to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = -1.$$

$\therefore Rf'(0) = 1$ and $Lf'(0) = -1$. $\therefore f$ is not derivable at $x = 0$.

Graphically, $f'(c)$ means the gradient of the curve $y = f(x)$ at the point $(c, f(c))$ and quantitatively $f'(c)$ means the rate of change of the function $f(x)$ at c, with respect to the independent variable x.

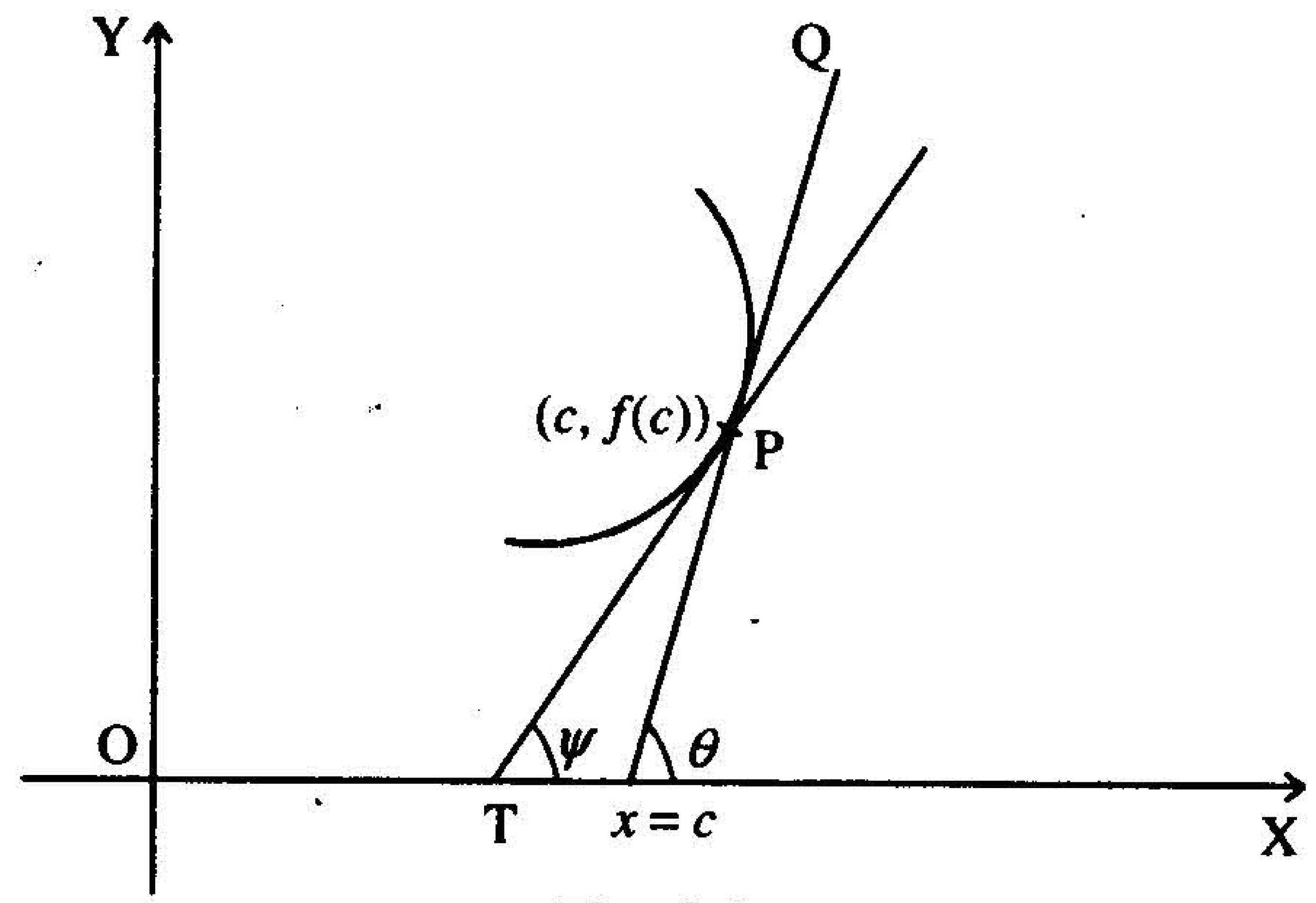

Fig. 8.1

§ 8.2 DERIVABILITY ON AN INTERVAL

Let $I = [a, b]$ be an interval and c be an interor point of I. If f is derivable at c, *i.e.*, $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists, derivable from the right at $x = a$ *i.e.*, $\lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$ exists and derivable from the left at $x = b$ *i.e.*, $\lim_{x \to b-} \frac{f(x)-f(b)}{x-b}$ exists, then $f(x)$ is said to be derivable on I.

If f is derivable on I, then its derivative at any point $x \in I$ is denoted by $f'(x)$.

Example 8.2.1. If f is continuous at $x = 0$, prove that $x f(x)$ is derivable at $x = 0$.

***Solution*:** Continuity of f at $x = 0$ implies that $\lim\limits_{x \to 0} f(x) = f(0)$

Let $\phi(x) = x f(x)$ then $\phi(0) = 0$.

Now $\lim\limits_{x \to 0} \dfrac{\phi(x) - \phi(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x f(x) - 0}{x} = \lim\limits_{x \to 0} f(x) = f(0)$.

$\therefore$ $\phi(x)$ is derivable at $x = 0$, and the derivative is $f(0)$.

Example 8.2.2. If f is discontinuous at c, prove that it is non derivable at c.

***Solution*:** If the function were derivable at $x = c$, then by Th. 8.1.1. the function would have been continous at c. But it is given that the function is discontinuous at c. Hence the function is not derivable at c.

Example 8.2.3. Show that the function $f(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$, but not derivable at $x = 0$.

***Solution*:** Choose $\varepsilon > 0$.

Here, $|f(x) - f(0)| = |x \sin\frac{1}{x} - 0| = |x|\,|\sin\frac{1}{x}| \leq |x| < \varepsilon$

if $|x - 0| = |x| < \delta\ (= \varepsilon)$. $\therefore$ $f(x)$ is continuous at $x = 0$.

Now, $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x \sin\frac{1}{x}}{x} = \lim\limits_{x \to 0} \sin\dfrac{1}{x}$

By Cauchy's condition for existence of limit, $\lim\limits_{x \to 0} \sin\dfrac{1}{x}$ does not exist.

[We chose $x_1 = \dfrac{1}{(4n+1)\frac{\pi}{2}}$ and $x_2 = \dfrac{1}{(4n-1)\frac{\pi}{2}}$ sufficiently close to 0 for suitably chosen large n.

$|f(x_1) - f(x_2)| = |\sin (4n + 1)\dfrac{\pi}{2} - \sin (4n - 1)\dfrac{\pi}{2}| = 2 \nless \varepsilon$

when ε is chosen arbitrarily.]

$\therefore$ $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist. Hence $f(x)$ is not derivable at $x = 0$.

[redacted] If $f(x) = \begin{cases} \sin x \sin\left(\frac{1}{\sin x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Show that $f(x)$ is continuous at $x = 0$, but not derivable at that point.
(C.H. 198

***Solution*:** Let us put $t = \sin x$

$\therefore$ $t = 0$ when $x = 0$, and $t \neq 0$ when $x \neq 0$ in a *nbd* of $x = 0$.

$\therefore$ the given function then becomes a function of t. We write

$$\phi(t) = \begin{cases} t \sin\frac{1}{t} & \text{when } t \neq 0 \\ 0 & \text{when } t = 0. \end{cases}$$

By Ex 8.2.3 above, $\phi(t)$ is continuous at $t = 0$ but not derivable at $t = 0$
$\therefore$ $f(x)$ is continuous at $x = 0$, but not derivable at $x = 0$.

[redacted] Let $f(x) = \begin{cases} 1 & \text{for } x < 0 \\ 1 + \sin x & \text{for } 0 \leq x < \frac{\pi}{2} \\ 2 + (x - \frac{\pi}{2})^2 & \text{for } x \geq \frac{\pi}{2} \end{cases}$

Show that $f'(x)$ exists at $x = \dfrac{\pi}{2}$, but it does not exist at $x = 0$. **(C.H. 199**

***Solution*:** Here $f(0) = 1$ and $f\left(\dfrac{\pi}{2}\right) = 2$.

$\lim\limits_{x \to 0-} \dfrac{f(x) - f(0)}{x - 0} = 0$ [for, $f(x) - f(0) = 0$ always in $(-\delta, 0)$]

$\therefore$ $f'(0 - 0) = 0$.

$\lim\limits_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0+} \dfrac{1 + \sin x - 1}{x} = \lim\limits_{x \to 0+} \dfrac{\sin x}{x} = 1$.

$\therefore$ $f'(0 + 0) = 1 \neq f'(0 - 0)$

$\therefore$ $f'(0)$ does not exist.

Also, $\lim\limits_{x \to \frac{\pi}{2}-} \dfrac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}} = \lim\limits_{h \to 0+} \dfrac{1 + \sin\left(\frac{\pi}{2} - h\right) - 2}{-h}$

[we put $x = \dfrac{\pi}{2} - h$ $(h > 0)$ $\therefore x \to \dfrac{\pi}{2} \Rightarrow h \to$

$= \lim\limits_{h \to 0+} \dfrac{\cos h - 1}{-h} = 0$.

In order to calculate $\lim\limits_{x \to \frac{\pi}{2}+} \dfrac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}}$, we put $x = \dfrac{\pi}{2} + h$ $(h > 0)$.

Then, $\lim\limits_{x \to \frac{\pi}{2}+} \dfrac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}} = \lim\limits_{h \to 0+} \dfrac{2 + h^2 - 2}{h} = \lim\limits_{h \to 0+} h = 0.$

$\therefore\ f'\left(\dfrac{\pi}{2} - 0\right) = f'\left(\dfrac{\pi}{2} + 0\right) = 0.$ $\therefore$ $f(x)$ is derivable at $x = \dfrac{\pi}{2}$.

Example 8.2.6. Show that the function g defined by

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

is differentiable everywhere, but the derived function is not continuous at $x = 0$.

(C.H. 1993, 1998)

***Solution*:** When $x \neq 0$, $g'(x) = 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$.

Again, $\lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^2 \sin \frac{1}{x}}{x} = \lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0.$

Because, $\left| x \sin \dfrac{1}{x} - 0 \right| = |x| \left| \sin \dfrac{1}{x} \right| \leq |x| < \varepsilon$ when $|x - 0| < \delta\ (= \varepsilon)$

$\therefore$ $g(x)$ is derivable at $x = 0$. $\therefore$ $g(x)$ is derivable for all $x \in \mathbb{R}$

$$\therefore\ g'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

Now, $\lim\limits_{x \to 0} \cos \dfrac{1}{x}$ does not exist by Cauchy's condition for existence of limit of a function.

[We choose $\varepsilon = 0.5$ and $x_1 = \dfrac{1}{2n\pi}$; $x_2 = \dfrac{1}{(2n+1)\pi}$ sufficiently close to 0 for large values of n. Then, $|f(x_1) - f(x_2)| = |\cos 2n\pi - \cos (2n + 1)\pi| = 2 \nless 0.5$].

$\therefore$ $\lim\limits_{x \to 0} g(x)$ does not exist. $\therefore$ $g'(x)$ is not continuous at the origin.

Example 8.2.7. A real-valued function f is defined as follows :

$$f(x) = \begin{cases} \frac{1}{q} \text{ when } x = \frac{p}{q}, p \text{ and } q \text{ are integers and } (p, q) = 1, (q > 0). \\ 0 \text{ when } x = 0 \text{ or } x \text{ is irrational} \end{cases}$$

Discuss derivability of $f(x)$ at $x = 0$.

***Solution*:** Here, $f(0) = 0$. $\therefore$ $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{f(x)}{x}$.

We are to show that $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$ does not exist. We consider a sequence $\{x_n\}_n$ where x_n is rational of the form $x_n = \dfrac{1}{n}$.

$\therefore$ $\lim\limits_{n \to \infty} x_n = 0$.

Now, $\lim\limits_{n \to \infty} \dfrac{f(x_n)}{x_n} = \lim\limits_{n \to \infty} n.\dfrac{1}{n} = 1.$

Again let $\{y_n\}_n$ be a sequence irrational points converging to 0.

$\therefore$ $\lim\limits_{n \to \infty} \dfrac{f(y_n)}{y_n} = \lim\limits_{n \to \infty} \dfrac{0}{y_n} = 0.$

$\therefore$ By sequential criterion $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$ does not exist.

$\therefore$ f is not derivable at $x = 0$.

Example 8.2.8. If $f(x) = |x|$ and $g(x) = 2|x|$, show that $f(x)$ and $g(x)$ are not derivable at $x = 0$. But $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)}$ exists and is equal to $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)}$.

***Solution*:** Here

$$f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0. \end{cases} \qquad g(x) = 2f(x).$$

$\lim\limits_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0} = 1.$ $\lim\limits_{x \to 0-} \dfrac{f(x) - f(0)}{x - 0} = -1.$

$\therefore$ $f(x)$ is not derivable at $x = 0$. So is $g(x)$.

But $\lim\limits_{x \to 0-} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0-} \dfrac{1}{2} = \dfrac{1}{2} = \lim\limits_{x \to 0+} \dfrac{f(x)}{g(x)}$

$$f'(x) = \begin{cases} 1 & \text{when } x > 0 \\ -1 & \text{when } x < 0 \end{cases} \quad g'(x) = \begin{cases} 2 & \text{when } x > 0 \\ -2 & \text{when } x < 0 \end{cases}$$

$$\therefore \lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x \to 0-} \frac{f'(x)}{g'(x)} = \lim_{x \to 0-} \left(\frac{1}{2}\right) = \frac{1}{2}$$

$\therefore \lim_{x \to 0} \frac{f'(x)}{g'(x)}$ exists and is equal to $\frac{1}{2}$. $\therefore \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{2}$.

Example 8.2.9. Show that, if $f(x) = \begin{cases} x^m \sin\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

$f(x)$ is derivable at $x = 0$. Also determine m when $f'(x)$ is continuous at $x = 0$.

Solution: $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^m \sin\frac{1}{x}}{x} = \lim_{x \to 0} x^{m-1} \sin \frac{1}{x}$

$$= \lim_{x \to 0} x^{m-2} \left(x \sin \frac{1}{x}\right) = \lim_{x \to 0} x^{m-2} . \lim_{x \to 0} \left(x \sin \frac{1}{x}\right) = 0.$$

$\therefore$ $f'(0)$ exists and $f'(0) = 0$

When $x \neq 0$. $f'(x) = m\, x^{m-1} \sin\frac{1}{x} - x^{m-2} \cos\frac{1}{x}$.

Thus $f'(x) = \begin{cases} m\, x^{m-1} . \sin\frac{1}{x} - x^{m-2} \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

Now, $\lim_{x \to 0} x^{m-1} \sin \frac{1}{x} = 0$ and $\lim_{x \to 0} x^{m-2} \cos\frac{1}{x}$ exists if $m > 2$ and the limiting value is then 0.

$\therefore \lim_{x \to 0} f'(x) = f'(0)$ when $m > 2$.

Thus $f'(x)$ is continnous when $m > 2$.

Example 8.2.10. If $f(x) = \begin{cases} \dfrac{x\left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

how that $f(x)$ is not derivable at $x = 0$.

Solution: $\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$.

$$= \lim_{x \to 0+} \frac{1 - e^{-\frac{2}{x}}}{1 + e^{-\frac{2}{x}}} \quad \left[\begin{array}{l} \frac{1}{x} \to +\infty \text{ as } x \to 0+ \\ e^{-\frac{1}{x}} \to 0 \text{ as } x \to 0+ \end{array}\right]$$

$$= 1.$$

$\therefore$ $Rf'(0) = 1$.

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{e^{\frac{2}{x}} - 1}{e^{\frac{2}{x}} + 1} = -1 \quad \left[\begin{array}{l} \frac{1}{x} \to -\infty \text{ as } x \to 0- \\ \text{and } e^{\frac{1}{x}} \to 0 \text{ as } x \to 0- \end{array}\right]$$

$\therefore$ $Lf'(0) = -1$. $\quad\therefore$ $f(x)$ is not derivable at $x = 0$.

Example 8.2.11. Show that $f(x) = x\,|\,x\,|$ is derivable at the origin.

Solution: Here, $\frac{f(x) - f(0)}{x - 0} = \frac{x|x|}{x} = |\,x\,|$

$\therefore \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} |\,x\,| = 0$. $\quad\therefore$ $f(x)$ is derivable at the origin.

Example 8.2.12. A function f is defined in $(-1, 1)$ by

$$f(x) = \begin{cases} x^p \sin\frac{1}{x^q} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

Prove that if $0 < q < p - 1$, f' is continuous at $x = 0$ and if $0 < p - 1 \leq q$, f' is discontinuous at $x = 0$.

Solution: When $x \neq 0$ $f'(x) = px^{p-1} \sin \frac{1}{x^q} - x^p . \cos\left(\frac{1}{x^q}\right)\frac{q}{x^{q+1}}$

$$= px^{p-1} \sin \frac{1}{x^q} - x^{p-q-1} . q \cos\left(\frac{1}{x^q}\right)$$

Also, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x^{p-1} \sin \frac{1}{x^q} = \lim_{x \to 0} x^{p-q-1} x^q \sin\left(\frac{1}{x^q}\right)$

$= 0 \quad$ if $p - q - 1 > 0$ and $q > 0$.

$$\therefore f'(x) = \begin{cases} px^{p-1} \sin\frac{1}{x^q} - qx^{p-q-1} \cos\left(\frac{1}{x^q}\right) & \text{when } x \neq 0 \\ 0 \quad \text{if } p - q - 1 > 0, \quad q > 0 & \text{when } x = 0 \end{cases}$$

$\therefore$ when $p - q - 1 > 0$ and $q > 0$, $\lim_{x \to 0} f'(x) = 0 = f(0)$

i.e., $f'(x)$ is continuous at $x = 0$.

When $p - q - 1 \le 0$, $\lim_{x \to 0} f'(x)$ does not exits.

$\therefore f'(x)$ is discontinnous at $x = 0$.

Example 8.2.13. Prove that the function f defined on $\mathbb{R}$ by

$$f(x) = |x - 1| + 2|x - 2| + 3|x - 3|$$

is continuous but not derivable at $x = 1, 2, 3$.

Solution: $$f(x) = \begin{cases} 1 - x + 2(2-x) + 3(3-x) = 14 - 6x & \text{in } -\infty < x < 1 \\ x - 1 + 2(2-x) + 3(3-x) = 12 - 4x & \text{in } 1 \le x < 2 \\ x - 1 + 2(x-2) + 3(3-x) = 4 & \text{in } 2 \le x < 3 \\ x - 1 + 2(x-2) + 3(x-3) = 6x - 14 & \text{in } x \ge 3. \end{cases}$$

In $(-\infty, 1)$, $f'(x) = -6$.

In $1 < x < 2$, $f'(x) = -4$

In $2 < x < 3$, $f'(x) = 0$.

For $x > 3$ $f'(x) = 6$.

$$\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{14 - 6x - 8}{x - 1} = \lim_{x \to 1-} \frac{-6(x-1)}{x - 1} = -6$$

$$\lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{12 - 4x - 8}{x - 1} = \lim_{x \to 1+} \frac{4 - 4x}{x - 1} = -4.$$

$\therefore$ $Rf'(1) \ne Lf'(1)$. $\quad \therefore f(x)$ is not derivable at $x = 1$.

Similarly, $$\lim_{x \to 2-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2-} \frac{12 - 4x - 4}{x - 2} = -4.$$

$$\lim_{x \to 2+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2+} \frac{4 - 4}{x - 2} = 0.$$

$\therefore Rf'(2) \ne Lf'(2)$ or, $f(x)$ is not derivable at $x = 2$.

$$\lim_{x \to 3+} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3+} \frac{6x - 14 - 4}{x - 3} = 6.$$

$$\lim_{x \to 3-} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3-} \frac{4 - 4}{x - 3} = 0.$$

$\therefore Rf'(3) \ne Lf'(3)$ *i.e.* $f(x)$ is not derivable at $x = 3$.

$\therefore$ The function is not derivable at $x = 1, 2, 3$, but derivable elsewhere.

Example 8.2.14. Let $f(x) = \begin{cases} x\left\{1 + \frac{1}{3}\sin(\log x^2)\right\} & \text{when } x \ne 0 \\ 0 & \text{when } x = 0. \end{cases}$

Show that $f(x)$ is continuous at $x = 0$ but not derivable there.

Solution: We have $f(0+0) = \lim_{h \to 0+} h\left\{1 + \frac{1}{3}\sin(\log h^2)\right\} = \lim_{h \to 0+} \left\{h + \frac{h}{3}\sin(\log h^2)\right\}$

$= 0$ $\quad$ [since $|\sin(\log h^2)| \le 1$].

Also, $f(0 - 0) = \lim_{h \to 0-} h\left\{1 + \frac{1}{3}\sin(\log h^2)\right\} = 0.$

$f(0) = 0 = \lim_{x \to 0} f(x)$ $\quad \therefore f(x)$ is continuous at $x = 0$.

Now, $$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x\left\{1 + \frac{1}{3}\sin(\log x^2)\right\}}{x} = \lim_{x \to 0+} \left\{1 + \frac{1}{3}\sin(\log x^2)\right\}$$

Since $\sin(\log x^2)$ oscillates between -1 and 1 as $x \to 0$,

$\lim_{x \to 0} \sin(\log x^2)$ does not exist. $\quad \therefore$ $Rf'(0)$ does not exist.

Similarly, $\lim_{x \to 0-} \frac{f(x) - f(0)}{x}$ does not exist.

$\therefore$ $Lf'(0)$ also does not exist. Thus f is not differentiable at $x = 0$.

Example 8.2.15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2 \sin\frac{1}{x^2}$ when $x \ne 0$

$= 0$ $\quad$ for $x = 0$.

Find the domain of f'. Is f' continuous there?

Solution: When $x \ne 0$. $f'(x) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$.

Now, $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x^2}}{x} = \lim_{x \to 0} x \sin\frac{1}{x^2} = 0.$$

$\therefore f(x)$ is differentiable at $x = 0$.

$$\therefore f'(x) = \begin{cases} 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2} & \text{when } x \ne 0 \\ 0 & \text{when } x = 0. \end{cases}$$

$\therefore f$ is differentiable on $\mathbb{R}$.

But $\lim\limits_{x \to 0} \frac{1}{x} \cos \frac{1}{x^2}$ does not exist.

Therefore, $f'(x)$ is not continnous at $x = 0$. In fact, $x = 0$ is a point of infinite discontinuity of $f'(x)$.

Example 8.2.16. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$, if x is rational
$= 0$, if x is irrational.

Show that f is differentiable only at 0.

Solution: Here, $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{x^2}{x} = 0$ [when x is rational]

Also $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{0}{x} = 0.$ [when x is irrational]

$\therefore \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists and is equal to 0. $\therefore f'(0) = 0$.

For any other point $\alpha \neq 0$, let α be a rational point

$\therefore \lim\limits_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = \lim\limits_{x \to \alpha} \phi(x)$, say.

Let us consider a sequence $\{x_n\}_n$ converging to α, where x_n is rational.

$\therefore f(x_n) = x_n^2$ and $\lim\limits_{n \to \infty} x_n = \alpha$

Now, $\lim\limits_{n \to \infty} \phi(x_n) = \lim\limits_{n \to \infty} \frac{f(x_n) - f(\alpha)}{x_n - \alpha} = \lim\limits_{n \to \infty} \frac{x_n^2 - \alpha^2}{x_n - \alpha}$

$= \lim\limits_{n \to \infty} (x_n + \alpha) = 2\alpha.$

Again, let $\{y_n\}_n$ be sequence of irrational points converging to α, then $f(y_n) = 0$ and $\lim\limits_{n \to \infty} y_n = \alpha$.

$\therefore \lim\limits_{n \to \infty} \phi(y_n) = \lim\limits_{n \to \infty} \frac{f(y_n - \alpha^2)}{y_n - \alpha} = \lim\limits_{n \to \infty} \frac{-\alpha^2}{y_n - \alpha}$ which is not finite.

$\therefore \lim\limits_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha}$ = does not exist.

$\therefore f(x)$ is not derivable at $x = \alpha$ when α ($\neq 0$) is a rational point.

Similarly, $f(x)$ is not derivable at any irrational point.

Thus $f(x)$ is derivable at $x = 0$ only.

Example 8.2.17. If $f'(c)$ exists prove that $\lim\limits_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = f'(c)$.

Solution: $f'(c)$ exists

$\therefore Rf'(c) = Lf'(c)$ and the common value is $f'(c)$.

Now, $\lim\limits_{h \to 0+} \left[\frac{f(c+h) - f(c-h)}{2h}\right]$

$= \lim\limits_{h \to 0+} \left[\frac{f(c+h) - f(c) + f(c) - f(c-h)}{2h}\right]$

$= \lim\limits_{h \to 0+} \frac{f(c+h) - f(c)}{2h} + \lim\limits_{h \to 0+} \frac{f(c) - f(c-h)}{2h}$

$= \frac{1}{2}Rf'(c) + \frac{1}{2} \lim\limits_{k \to 0-} \frac{f(c+k) - f(c)}{2k}$ $[h = -k]$

$= \frac{1}{2}Rf'(c) + \frac{1}{2}Lf'(c) = f'(c).$

Also, $\lim\limits_{h \to 0-} \frac{f(c+h) - f(c-h)}{2h}$

$= \lim\limits_{h \to 0-} \left[\frac{f(c+h) - f(c)}{2h} + \frac{f(c) - f(c-h)}{2h}\right]$

$= \lim\limits_{h \to 0-} \frac{f(c+h) - f(c)}{2h} + \lim\limits_{\lambda \to 0+} \frac{f(c+\lambda) - f(c)}{2\lambda}$ $[-h = \lambda]$

$= \frac{1}{2}Lf'(c) + \frac{1}{2}Rf'(c) = f'(c)$. $\therefore \lim\limits_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = f'(c)$.

Example 8.2.18. Show by an example that the result of example 8.2.17. holds even if $f'(c)$ fails to exist.

Solution: Let us consider $f(x) = |x - 1|$.

Now $f'(1)$ does not exist.

$\lim\limits_{h \to 0} \frac{f(1+h) - f(1-h)}{2h} = \lim\limits_{h \to 0} \frac{|h| - |-h|}{2h} = 0.$

§ 8.3 ALGEBRA OF DERIVATIVES

Theorem 8.3.1: Let f, g be two real valued functions defined on an interval I and $c \in I$. Then the following relations hold, whenever $f'(c)$ and $g'(c)$ exist.

(i) $f \pm g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.

(ii) fg is differentiable at c and $(fg)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$.

(iii) If $g(c) \neq 0$, $\frac{f}{g}$ is differentiable at c and $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)\,g(c) - f(c)\,g'(c)}{\{g(c)\}^2}$.

It is worth noting that the converse proposition of the theorems are not necessarily true. We cite examples.

(i) $f(x) = |x|$ and $g(x) = -|x|$, then $f(x) + g(x) = 0\ \forall x$ is derivable at $x = 0$ but $f'(0)$ and $g'(0)$ do not exist.

(ii) $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$, and $g(x) = 0\ \forall\ x \in \mathbb{R}$

so that $f(x)\, g(x)$ is derivable at $x = 0$. But $f(x)$ is not derivable at $x = 0$.

§ 8.4. SIGN OF DERIVATIVE AT A POINT

Definition : (Increasing and Decreasing Function at a Point)

Let f be defined on I. and c be an interior point of I, f is said to be *increasing* at c, if there exists a $\delta\ (> 0)$ such that

$f(x) > f(c)\ \forall\ x \in I$ satisfying $c < x < c + \delta$, and

$f(x) < f(c)\ \forall\ x \in I$ satisfying $c - \delta < x < c$.

Similarly, f is *decreasing* at c if there exists a $\delta\ (> 0)$ such that

$f(x) < f(c)\ \forall\ x \in I$ and satisfying $c < x < c + \delta$

and $f(x) > f(c)\ \forall\ x \in I$ and satisfying $c - \delta < x < c$.

If $I = [a, b]$, f is increasing at a if for a positive δ, $f(x) > f(a)$ whenever $a < x < a + \delta$ and f is decreasing at a if for a $\delta (> 0)$ $f(x) < f(a)$ for $a < x < a + \delta$.

Similarly, for the right end point b, $f(x)$ is increasing at b if there exist a $\delta (> 0)$ such that $f(x) < f(b)$ whenever $b - \delta < x < b$, and $f(x)$ is decreasing at b if for a positive δ, $f(x) > f(b)$ whenever $b - \delta < x < b$.

Theorem 8.4.1: Let a real-valued function f be differentiable at $c \in I$, the domain of definition of f,

then (i) f is increasing at c if $f'(c) > 0$

(ii) f is decreasing at c if $f'(c) < 0$.

Proof. f is differentiable at c. By definition,

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists and is equal to $f'(c)$.

If $f'(c)$ is > 0., there exists a *nbd* $N_\delta(c)$ of c

such that $\frac{f(x) - f(c)}{x - c} > 0$.

If $c - \delta < x < c$, $f(x) - f(c) < 0$ *i.e.,* $f(x) < f(c)$

and if $c < x < c + \delta$ $f(x) - f(c) > 0$ *i.e.,* $f(x) > f(c)$

$f(x)$ is increasing at c.

If $f'(c) < 0$, there exists a *nbd* $N_{\delta'}(c)$ of c

where $\frac{f(x) - f(c)}{x - c}$ is negative.

Now $f(x) - f(c) > 0$ when $c - \delta' < x < c$

and $f(x) - f(c) < 0$ when $c < x < c + \delta'$.

By what has been shown above it proves that $f(x)$ is decreasing at c.

If c is an end point of the interval I, slight modification in above argume will establish the result.

Note: The above conditions are sufficient but by no means necessary for a function be increasing or decreasing at a point. We consider the following examples.

$f(x) = x^3$ is increasing at $x = 0$, but $f'(0) = 0$.

Also $f(x) = -x^3$ is decreasing at $x = 0$, but $f'(0) = 0$.

Example 8.4.1. Show that $x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1 + x)}\ \ \forall\ x >$

Solution: Let $F(x) = \log(1 + x) - \left(x - \frac{x^2}{2}\right)$

$$F'(x) = \frac{1}{1 + x} - (1 - x) = \frac{x^2}{1 + x} > 0\ \ \forall\ x > 0.$$

$\therefore$ $F(x)$ is increasing for $x > 0$ *i.e.,* $F(x) > F(0) = 0$.

$$\therefore\ \log(1 + x) > x - \frac{x^2}{2}\ \ \forall\ x > 0.$$

Again, let $\phi(x) = x - \frac{x^2}{2(1 + x)} - \log(1 + x)$

$$\therefore\ \phi'(x) = 1 - \frac{x}{1+x} + \frac{x^2}{2(1+x)^2} - \frac{1}{x+1} = \frac{x^2}{2(1+x)^2} > 0.$$

$\therefore\ \phi(x)$ is also increasing for $x > 0$ *i.e.*, $\phi(x) > \phi(0) = 0$.

$$\therefore\ x - \frac{x^2}{2(1+x)} > \log(1+x)\ \ \forall\ x > 0.$$

$$\therefore\ x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}\ \ \forall\ x > 0.$$

Example 8.4.2. Prove that $\tan x > x$ when $0 < x < \frac{\pi}{2}$.

Solution: Let c be any real number such that $0 < c < \frac{\pi}{2}$.

Let $f(x) = \tan x - x\ \forall\ x \in [0, c]$.
f is continuous and derivable in $[0, c]$.
$f'(x) = \sec^2 x - 1 > 0$ when $0 < x < c$.
$\therefore\ f$ is strictly increasing in $[0, c]$. $\quad \therefore\ f(c) > f(0) = 0$.
$\therefore\ f(c)$ is positive *i.e.* $\tan x - x > 0$ when $x = c$. Since c is any real number such that $0 < c < \frac{\pi}{2}$.

$\therefore\ \tan x > x$ when $0 < x < \frac{\pi}{2}$.

Example 8.4.3. Prove that $\frac{2x}{\pi} < \sin x < x$ when $0 < x < \frac{\pi}{2}$. **(C.U. 1996)**

Solution: Let us consider the function $f(x)$ defined as

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}.$$

Here, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$.

$\therefore\ f$ is continuous at $x = 0$. Also, in $\left(0, \frac{\pi}{2}\right]$, $f(x)$ is continuous

$\therefore\ f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$.

When $x \neq 0$ $\quad f'(x) = \frac{(x - \tan x)\cos x}{x^2}$.

Now, in $0 < x < \frac{\pi}{2}$ $\tan x > x$ and $\cos x > 0$. $\qquad \therefore\ f'(x) < 0$.

$\therefore\ f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$.

$\therefore\ f(0) > f(x) > f\left(\frac{\pi}{2}\right)$ $\quad$ *i.e.*, $1 > \frac{\sin x}{x} > \frac{2}{\pi}$

or, $\frac{2x}{\pi} < \sin x < x$ when $0 < x < \frac{\pi}{2}$.

Example 8.4.4. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ in $0 < x < \frac{\pi}{2}$.

Solution: $\frac{\tan x}{x} - \frac{x}{\sin x} = \frac{\tan x\ \sin x - x^2}{x \sin x}$

We put $f(x) = \tan x \sin x - x^2$.

Let $0 < c < \frac{\pi}{2}$. In $[0, c]$, $f(x)$ is continuous and derivable.

$f'(x) = \sin x\,(1 + \sec^2 x) - 2x$.
We can not determine sign of $f'(x)$ in $[0, c]$.
$\therefore\ f''(x) = \cos x\,(\sec^2 x + 1) + 2\sec^2 x \tan x.\ \sin x - 2$

$$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin^2 x.\ \sec^3 x > 0 \text{ in } (0, c).$$

$\therefore\ f'(x) > f'(0)$ in $0 < x < c$.
$\qquad f'(x) > 0$ in $0 < x < c$.
Therefore $f(x)$ is increasing in $[0, c]$.
$f(x) > f(0) = 0$ implies $f(x) > 0$

or, $\tan x \sin x - x^2 > 0$ or, $\frac{\tan x}{x} > \frac{x}{\sin x}$ in $0 < x < \frac{\pi}{2}$.

Example 8.4.5. If $0 < x < 1$, show that $2x < \log\frac{1+x}{1-x} < 2x\left(1 + \frac{1}{3}\frac{x^2}{1-x}\right.$

Also deduce that $e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1 + \frac{1}{12n(n+1)}}$.

Solution: Let $f(x) = 2x - \log \dfrac{1+x}{1-x}$ in $[0, 1)$.

If $0 < c < 1$, then f is continuous on $[0, c]$ and derivable there.

Also $f'(x) = \dfrac{-2x^2}{1-x^2} < 0$ in $(0, c)$.

$\therefore$ f is decreasing in $(0, c)$ *i.e.,* $f(c) < f(0) = 0$.

or, $2x - \log \dfrac{1+x}{1-x} < 0$ when $x = c$.

Since $c \in (0, 1)$ it follows that $2x < \log \dfrac{1+x}{1-x}$ in $(0, 1)$.

Let again, $g(x) = \log \dfrac{1+x}{1-x} - 2x\left(1 + \dfrac{1}{3}\dfrac{x^2}{1-x^2}\right)$ in $[0, 1)$.

$g'(x) = -\dfrac{4}{3}\dfrac{x^2}{(1-x^2)^2} < 0.$ $\quad \therefore g(x) < g(0) = 0.$

or, $\log \dfrac{1+x}{1-x} < 2x\left(1 + \dfrac{1}{3}\dfrac{x^2}{1-x^2}\right)$, when $0 < x < 1$.

$\therefore 2x < \log \dfrac{1+x}{1-x} < 2x\left(1 + \dfrac{1}{3}\dfrac{x^2}{1-x^2}\right).$

In the above inequality, let us put $x = \dfrac{1}{2n+1}$.

$\therefore \dfrac{2}{2n+1} < \log \dfrac{n+1}{n} < \dfrac{2}{2n+1}\left(1 + \dfrac{1}{12n(n+1)}\right).$

or, $1 < \left(n + \dfrac{1}{2}\right) \log \dfrac{n+1}{n} < 1 + \dfrac{1}{12n(n+1)}$

or, $e < \left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}} < e^{1 + \frac{1}{12n(n+1)}}.$

Theorem 8.4.2 (DARBOUX'S THEOREM)

If a function f is derivable on a closed interval $[a, b]$, and $f'(a)$, $f'(b)$ are of opposite signs then there exists at least one point c in (a, b) such that $f'(c) = 0$.

(C.H. 1996, 2000)

Proof. Without any loss of generality, let us suppose that

$f'(a) < 0$ and $f'(b) > 0$.

Since f is derivable in $[a, b]$ it is continuous there. Therefore, f is bounded in $[a, b]$ and attains its bounds there.

Thus if m is $\inf\limits_{x \in [a, b]} f(x)$, there exists a point c in $[a, b]$ such that $m = f(c)$.

Now, $f'(a) < 0$ implies that f is decreasing at a.

Therefore, there exists a positive δ_1 such that

$f(x) < f(a)$, when $x \in (a, a + \delta_1)$

$\therefore c \neq a$, for then $f(x) < f(c) = m$, in $(c, c + \delta_1)$ would be a contradiction, since there can not exist values of $f(x) < m$.

Again, $f'(b) > 0$ implies that f is increasing at b.

Therefore, these exists a positive δ_2 such that

$f(x) < f(b)$ when $b - \delta_2 < x < b$.

$\therefore c \neq b$, by arguments given above.

$\therefore c \in (a, b)$, and we propose to show that $f'(c) = 0$.

If not, let $f'(c) > 0$, then $f(x)$ is increasing at c. There exists, therefore, a positive δ_3 such that $f(x) < f(c) = m$ in $c - \delta_3 < x < c$ which is a contradiction.

If $f'(c) < 0$, then $f(x)$ is decreasing at c, and there exists a δ_4 (> 0) such that

$f(x) < f(c) = m$ in $c < x < c + \delta_4$ which is again a contradiction.

Therefore, only conclusion is that $f'(c) = 0$.

Theorem 8.4.3 : (Intermediate Value Property of Derived Function)

If f be derivable in $[a, b]$ and $f'(a) \neq f'(b)$. Then $f'(x)$ will assume every value between $f'(a)$ and $f'(b)$ at least once in the interval (a, b).

If in particular, f' is monotone in (a, b), f' is continuous in (a, b).

(C.H. 2002)

Proof. Let $f'(a) < f'(b)$ and k lies between $f'(a)$ and $f'(b)$

i.e., $f'(a) < k < f'(b)$.

We construct the function $g(x) = f(x) - kx$ which is derivable in $[a, b]$ since $f(x)$ is so,

$g'(x) = f'(x) - k$.

Now, $g'(a) = f'(a) - k < 0$ and $g'(b) = f'(b) - k > 0$.

$\therefore g'(a)$ and $g'(b)$ are of opposite signs. Therefore, by Darboux's theorem there exists a point $c \in (a, b)$

such that $g'(c) = 0$ *i.e.,* $f'(c) = k$.

An independent proof is given below

Proof. Let $g(x) = f(x) - kx \; \forall \; x \in [a, b]$. Then $g(x)$ is derivable in $[a, b]$ and $g'(x) = f'(x) - k$.

$g'(a) = f'(a) - k$ and $g'(b) = f'(b) - k$ are of opposite signs. Let, for definiteness, $g'(a) > 0$ and $g'(b) < 0$.

There exist, δ_1 and $\delta_2 > 0$ such that

$$0 < h < \delta_1 \Rightarrow \left| \frac{g(a+h)-g(a)}{h} - g'(a) \right| < g'(a) = \varepsilon$$

$$\Rightarrow \frac{g(a+h)-g(a)}{h} > 0 \Rightarrow g(a) < g(a+h) \qquad \dots (1)$$

$$\text{and } 0 < h < \delta_2 \Rightarrow \left| \frac{g(b-h)-g(b)}{h} - g'(b) \right| < -g'(b)$$

$$\Rightarrow g(b) < g(b-h) \qquad \dots (2)$$

Since g is derivable on $[a, b]$, it is continuous there and its supremun is attained on $[a, b]$.

(1) and (2) imply that the supremum is attained neither at a nor at b. Hence there is a point $c \in (a, b)$ such that $g(c) = \sup g$.

We propose to show that $g'(c) = 0$.

If possible, let $g'(c) > 0$. Since g is derivable at c there exists a δ_3 (> 0) such that for $\in = g'(c)$

$$0 < h < \delta_3 \Rightarrow \left| \frac{g(c+h)-g(c)}{h} - g'(c) \right| < g'(c)$$

$$\Rightarrow g(c) < g(c+h)$$

This contradicts that $g(c) \neq \sup g$ on $[a, b]$.

$\therefore g'(c) \not> 0$. Similarly, $g'(c) \not< 0 \quad \therefore g'(c) = 0$ or $f'(c) = k$.

Particular case

Let us assume that f' has a discontinuity at some point c in (a, b). We choose a closed sub-interval $[\alpha, \beta]$ of (a, b) such that $[\alpha, \beta]$ contains c as an interior point. Since f' is monotonic on $[\alpha, \beta]$ f' has a jump discontinuity. Hence f' does not attain some values between $f'(\alpha)$ and $f'(\beta)$ contradicting the intermediate value theorem.

$\therefore f'$ is continuous on (a, b).

> **Corollary I :** If f is derivable on an interval I, then $f'(I)$ is either an interval or a singleton set.

Proof. Let $J = f'(I)$. Let $p_1 \neq p_2 \in J$, then $x_1 \neq x_2 \in I$ such that $f'(x_1) = p_1$ and $f'(x_2) = p_2$. Let $[x_1, x_2] \subset I$ and $p_1 < p < p_2$. Then there exists $c \in (x_1, x_2)$ such that $f'(c) = p$.

$\therefore p \in J$. Thus if p_1, p_2 are distinct members of J, then every member between p_1 and p_2 belongs to J.

Hence J is an interval. If J does not contain at least two distinct members then J is a singleton set.

Note : $S \subset \mathbb{R}$ is an interval if for any $x_1 < x_2$, where $x_1, x_2 \in S$, $[x_1, x_2] \subset S$.

> **Corollary 2 :** If $f'(x) \neq 0 \; \forall x \in (a, b)$, then $f'(x)$ retains the same sign

Proof. If possible, let $x_1, x_2 \in (a, b)$ such that $f'(x_1) f'(x_2) < 0$. *i.e.* $f'(x_1)$ and $f'(x_2)$ are of opposite sign. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This contradicts the hypothesis that $f'(x) \neq 0$ in (a, b). Hence $f'(x)$ retains the same sign in (a, b).

Example 8.4.6. Show that the function f defined by $f(x) = [x]$ in $[0, x]$ is not the derivative of any function. **(C.H. 2000)**

Solution: If possible, let there is a function $g(x)$ such that $g'(x) = f(x) = [x]$ in $[0, x]$

Now $g'(0) = 0$ and $g'(1) = 1$.

$\therefore$ $g'(x)$ will assume every value between 0 and 1 (by above theorem).

But in $[0, 1]$ $g'(x) = f(x) = [x]$ has the value either 0 and 1.

Infact, $[x] = 0$ when $0 \leq x < 1$

$= 1$ when $x = 1$.

Hence, we arrive at a contradiction, and, therefore, there is no function $g(x)$ whose derivative is $f(x)$.

Example 8.4.7. If a function $f(x)$ be defined as $f(x) = 0$ when $x \neq 0$ and $f(x) = 1$ when $x = 0$, there exists no function $g(x)$ such that $g'(x) = f(x)$.

Solution: If possible, let there be a function $g(x)$ such that

$$g'(x) = f(x) = \begin{cases} 0 & \text{when } x \neq 0 \\ 1 & \text{when } x = 0. \end{cases}$$

$g(x)$ is differentiable in any interval $[0, \alpha]$, then $g'(0) = 1$ and $g'(\alpha) = 0$. Then by intermediate value property of derived function $g'(x)$ assumes every value between 0 and 1; but by definition of $f(x)$, $g'(x)$ assumes no value between 0 and 1 in any interval $[0, \alpha]$.

$\therefore$ no such function $g(x)$ exists such that $g'(x) = f(x)$.

Example 8.4.8. Show that there does not exists a function ϕ such that $\phi'(x) = f(x)$ on $[0, 2]$ where $f(x) = [x] \quad x \in [0, 2]$.

***Solution*:** Here $f(x) = 0$ when $0 \le x < 1$

$= 1$ when $1 \le x < 2$

$= 2$ when $x = 2$.

If possible $\phi'(x) = f(x) = \phi$ is differentiable in $[0, 2]$ Then $\phi'(0) = 0$ and $\phi'(2) = 2$.

$\therefore$ By intermediate value property of derived function, $\phi'(x)$ assumes every value between 0 and 2 but since $f(x)$ has no other value except 1, between 0 and 2, $\phi'(x)$ also does not assume any other value except 1 between 0 and 2 in $[0, 2]$.

$\therefore$ no such function $\phi(x)$ does exist such that $\phi' = f$.

Theorem 8.4.4: If f is differentiable in (a, b) and at a point $c \in (a, b)$ $\lim_{x \to c} f'(x) = l$ exists, then f' is continuous at c. **(C.H., 1998)**

Proof. $\lim_{x \to c} f'(x) = l \Rightarrow \lim_{x \to c-} f'(x) = l = \lim_{x \to c+} f'(x)$.

$f'(x)$ exists at $x = c \in (a, b)$

and let $f'(c) > l$. We choose ε such that

$$0 < \varepsilon < f'(c) - l \ \ i.e., \ \ l + \varepsilon < f'(c).$$

Now, since $\lim_{x \to c-} f'(x) = l$, there exists a positive δ such that

$|f'(x) - l| < \varepsilon$ when $x \in (c - \delta, c)$ for all x in (a, b)

or, $l - \varepsilon < f'(x) < l + \varepsilon \ \ \forall \ x \in (c - \delta, c) \cap (a, b)$ ... (1)

Let p be a point in $(c - \delta, c) \cap (a, b)$

$\therefore \ l - \varepsilon < f'(p) < l + \varepsilon < f'(c)$

$\therefore$ by intermediate value property of derived function there exists a point $\xi \in (p, c)$ such that $f'(\xi) = l + \varepsilon$

But from (1) $f'(\xi) < l + \varepsilon$ for $\xi \in (c - \delta, c)$.

Thus we arrive at a contradiction. $\therefore f'(c) \ngtr l$.

Now, let $f'(c) < l$. We choose ε such that $0 < \varepsilon < l - f'(c)$.

Since $\lim_{x \to c-} f'(x) = l$, there exists a δ_1 (> 0) such that

$|f'(x) - l| < \varepsilon$ when $x \in (c - \delta_1, c)$

$\therefore l - \varepsilon < f'(x) < l + \varepsilon$ when $x \in (c - \delta_1, c) \cap (a, b)$ ... (2)

Let $q \in (c - \delta_1, c) \cap (a, b)$, then $f'(c) < l - \varepsilon < f'(q) < l + \varepsilon$.

$\therefore$ There exists a point $\eta \in (q, c)$ such that $f(\eta) = l - \varepsilon$

But by (2) $f'(\eta) > l - \varepsilon$, a contradiction.

$\therefore f'(c) < l$ is not possible. Hence, $\lim_{x \to c-} f'(x) = f'(c)$.

Exactly in the same way we can prove that $\lim_{x \to c+} f'(x) = f'(c)$.

$\therefore \lim_{x \to c} f'(x) = f'(c)$ proves that $f'(x)$ is continuous at $x = c$.

Corollary : A derived function f' on I can not have a jump discontinuity on I.

Example 8.4.9. If f is a real-value function defined on $[-1, 1]$ by

$f(x) = 0$ when $-1 \le x \le 0$

$= 1$ when $0 < x \le 1$.

Does there exist a function ϕ such that $\phi'(x) = f(x)$ on $[-1, 1]$?

***Solution*:** If possible, let there exists a function ϕ on $[-1, 1]$ such that $\phi'(x) = f(x)$

Then ϕ is differentiable and $\phi'(-1) = 0$, $\phi'(1) = 1$.

By intermediate value property of derived function $\phi'(x)$ assumes every value between 0 and 1.

But $\phi'(x) = f(x) = 0$ when $-1 \le x \le 0$

$= 1$ when $0 < x \le 1$.

$\therefore$ no such function $\phi(x)$ does exist such that $\phi'(x) = f(x)$ on $[-1, 1]$.

Example 8.4.10. Show that there does not exist a function ϕ such that $\phi'(x) = f(x)$ on $[0, 2]$, where $f(x) = x - [x]$

***Solution*:** Here $f(x) = x$ for $0 \le x < 1$

$= x - 1$ for $1 \le x < 2$

$= 0$ for $x = 2$.

If possible, let there exist a function ϕ such that $\phi'(x) = f(x)$

Now, $\lim_{x \to 1-} f(x) = 1$. $\lim_{x \to 1+} f(x) = 0 = f(1)$

$f(x)$ has jump discontinuity at $x = 1$.

But by Theorem 8.4.4., a derived function can not have a jump discontinuity.

$\therefore$ no such function $\phi(x)$ exists.

Example 8.4.11. Prove that if the mapping f on $\mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then $|f|$ is differentiable at a, provided $f(a) \ne 0$. Is the result true for $f(a) = 0$?

***Solution*:** The mapping f on $\mathbb{R}$ is differentiable at $a \in \mathbb{R}$ then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is equal to $f'(a)$. Since f is differentiable at a, it is continuous at a. It is given that $f(a) \neq 0$. Let $f(a) > 0$. then there exists a $\delta\ (> 0)$ such that $f(x) > 0$ when $N_\delta(a)$ (neighbourhood property of a continuous function).

$\therefore\ |f(x)| - |f(a)| = f(x) - f(a)$, when $|x - a| < \delta$;

and therefore, $\lim\limits_{x \to a} \dfrac{|f|(x) - |f|(a)}{x - a}$

$$= \lim_{x \to a} \frac{|f(x)| - |f(a)|}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Similarly, if $f(a) < 0$, we have

$$\lim_{x \to a} \frac{|f|(x) - |f|(a)}{x - a} = -\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = -f'(a).$$

$\therefore\ |f|$ is differentiable at $x = a$.

The result is, in general, not true if $f(a) = 0$. We consider the function $f(x) = |x|$.

Here, $f(0) = 0$ but $f'(0)$ does not exist.

Example 8.4.12. Let $f(x + y) = f(x) + f(y)\ \forall\ x, y \in \mathbb{R}$. Prove that f is derivable on $\mathbb{R}$, if f is derivable at one point of R.

Solution: Let f be derivable at $a \in \mathbb{R}$.

Then $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists and is equal to $f'(a)$

or, $\lim\limits_{h \to 0} \dfrac{f(a) + f(h) - f(a)}{h} = f'(a)$ or, $\lim\limits_{h \to 0} \dfrac{f(h)}{h} = f'(a)$.

Now, at any other point $c \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c) + f(h) - f(c)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = f'(a).$$

$\therefore\ f$ is derivable at any other point $c \in \mathbb{R}$. $\therefore\ f$ is derivable on $\mathbb{R}$.

Theorem 8.4.5: If f is derivable at $c \in (a, b)$ then there exists a $\delta\ (> 0)$ and a positive number M, such that $|f(x) - f(c)| < M\,|x - c|$ when $x \in N'_\delta(c)$. Hence f is continuous at c. **(C.H.)**

Proof. f is derivable at c implies that

$\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists and is equal to $f'(c)$.

Therefore, for a given $\varepsilon\ (> 0)$ there exists a $\delta\ (> 0)$

Such that $\left|\dfrac{f(x) - f(c)}{x - c} - f'(c)\right| < \varepsilon$ when $x \in N'_\delta(c)$.

Let us choose $\varepsilon = 1$, then

$$|f(x) - f(c)| = \left|\frac{f(x) - f(c)}{x - c}\right| |x - c| = \left|\frac{f(x) - f(c)}{x - c} - f'(c) + f'(c)\right| |x - c|$$

$$\leq |x - c| \left\{\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| + |f'(c)|\right\}$$

$$< |x - c|\,\{1 + |f'(c)|\} = M\,|x - c| \quad (M = 1 + |f'(c)|)$$

$\therefore\ |f(x) - f(c)| < M\,|x - c|$ when $x \in N'_\delta(c)$.

Now, we choose $\delta = \dfrac{\varepsilon}{M}$. Then

$|f(x) - f(c)| < M\,|x - c| < \varepsilon$, when $x \in N_\delta(c)$

This shows that $f(x)$ is continuous at $x = c$.

MORE ILLUSTRATIVE EXAMPLES

Example 1. If f and g are derivable at c and $f(c) \neq g(c)$ prove that each of the functions $h(x) = \max\{f, g\}$, and $k(x) = \min\{f, g\}$ is derivable at c. Discuss also the case when $f(c) = g(c)$.

Solution: $f(c) \neq g(c)\ \therefore (f - g)(c) \neq 0\ \therefore$ by *Ex.* 8.4.11, $|f - g|$ is derivable at c.

Again, since f and g are derivable at c, $f \pm g$ are derivable at c.

Now, $h(x) = \dfrac{1}{2}(f + g) + \dfrac{1}{2}|f - g|$ is derivable at c.

Also, $k(x) = \dfrac{1}{2}(f + g) - \dfrac{1}{2}|f - g|$ is derivable at c.

When $(f - g)(c) = 0$, $|f - g|$ may not be derivable at c. Hence $h(x)$ and $k(x)$ may not be derivable at c.

Example 2. Let $f(x + y) = f(x)\,f(y)\ \forall\ x, y \in \mathbb{R}$, and f be derivable at $x = 0$. If $f'(0) = l$, and $f(c) = m \neq 0$ for $c \neq 0$, prove that f is derivable at c and $f'(c) = lm$.

Solution: f is derivable at $x = 0$, $f(0) = 1$.

$\therefore\ \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = l$. or, $\lim\limits_{h \to 0} \dfrac{f(h) - 1}{h} = l$.

Now, $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = \lim\limits_{h\to 0} \dfrac{f(c).f(h)-f(c)}{h}$

$= \lim\limits_{h\to 0} f(c)\left\{\dfrac{f(h)-1}{h}\right\} = f(c) \times \lim\limits_{h\to 0} \dfrac{f(h)-1}{h} = lm.$

Thus f has derivative at c.

Example 3. If f is derivable on $[a, b]$, $f(a) = f(b) = 0$ and $f(x) \neq 0$ for any $x \in (a, b)$, then prove that $f'(a) f'(b) < 0$.

Solution: f is derivable on $[a, b]$ implies that f is continuous on $[a, b]$. f has right continuity at $x = a$, and left continuity at $x = b$.

Given that $f'(a)$ exists.

$\therefore \lim\limits_{h\to 0+} \dfrac{f(a+h)-f(a)}{h} = f'(a)$ or, $\lim\limits_{h\to 0+} \dfrac{f(a+h)}{h} = f'(a)$.

Similarly, $\lim\limits_{h\to 0+} \dfrac{f(b-h)}{h} = f'(b)$

Note that, $f'(a) \neq 0$ and $f'(b) \neq 0$ [since $f(x) \neq 0$ for any $x \in (a, b)$].

Now, $f'(a)$ and $f'(b)$ can not be of the same sign.

If $f'(a) > 0$ and $f'(b) > 0$ then f is increasing at a and b both

$\therefore f(a + h) > f(a) = 0$ and $f(b - h) < f(b) = 0$.

By continuity of f in $[a, b]$ there is a point $c \in (a + h, b - h) \subset [a, b]$ such that $f(c) = 0$ contradicting that $f(x) \neq 0$ in (a, b).

Similarly, $f'(a) < 0$ and $f'(b) < 0$ is also not posible. $\therefore f'(a) f'(b) < 0$.

EXERCISE 8

1. Let $f(x) = \begin{cases} x\tan^{-1}\left(\frac{1}{x}\right) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

examine if $f'(0)$ exists. **(C.H.)**

2. A function $f(x)$ is defined for $x > 0$ as follows.

$$f(x) = \begin{cases} 1-x^2 & \text{for} \quad 0 < x \leq 1 \\ \log x & \text{for} \quad 1 < x \leq 2 \\ \log 2 - 1 + \frac{1}{2}x & \text{for} \quad x > 2 \end{cases}$$

Obtain the domain of the derived function [$f'(1)$ and $f'(2)$ do not exist].

3. $f(x) = \{|x| - |x - 1|\}^2$. Discuss differentiability of $f(x)$ on $\mathbb{R}$.

4. Show that the function f defined on $\mathbb{R}$ by $f(x) = x^2 \cos\dfrac{1}{x}$ when $x \neq 0$, $f(0) = 0$ is derivable on R. What can be said about continuity of f' at $x = 0$? **[Not continuous]**

5. Is continuity at a point a sufficient condition for differentiability at that point? Justify. **[No]**

6. Let $f(x) = x(x-1)^2(x-2)^3 \sin\dfrac{1}{x} \sin\dfrac{1}{x-1} \sin\dfrac{1}{x-2}$ $\forall\, x \in \mathbb{R} \setminus \{0, 1, 2\}$ and

$f(x) = 0$ when $x = 0, 1, 2$.

Show that (i) $f'(x)$ does not exist at $x = 0$.

(ii) $f'(x)$ exists but not continuous at $x = 1$.

(iii) $f'(x)$ exists and is continuous at $x = 2$.

7. If $f(x + y) = f(x) + f(y)$ $\forall\, x, y \in \mathbb{R}$, prove that f is derivable on $\mathbb{R}$ if it is derivable at one point of $\mathbb{R}$.

8. Give example of function which is
   (i) continuous at a point, but not differentiable;
   (ii) continuous at a point, and differentiable there;
   (iii) neither continuous nor differentiable at a point;
   (iv) continuous at a point, differentiable there but the derived function is not continuous there.

9. If f is derivable on $[a, b]$, $f(a) = f(b) = 0$, and $f'(a) f'(b) > 0$, then show that there exists a point $c \in (a, b)$ such that $f(c) = 0$.

10. If $f(x) = \begin{cases} x^2 & \text{when } x \text{ is rational} \\ \sin x & \text{when } x \text{ is irrational}, \end{cases}$

show that f is derivable at $x = 0$ and $f'(0) = 1$.

11. A function f is defined as follows :

$f(x) = x^2$ when x is rational

$= 0$ when x is irrational

Show that f is differentiable only at $x = 0$.

12. Let $f(x + y) = f(x) f(y)$ $\forall\, x, y \in \mathbb{R}$. Prove that f is derivable on $\mathbb{R}$ if f is derivable at one point of $\mathbb{R}$.